Physics 513, Quantum Field Theory
Final Examination

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1. The Decay of a Scalar Particle

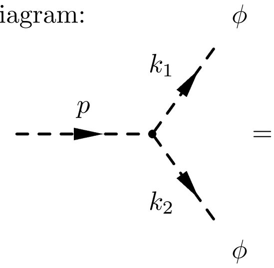
From the Lagrangian given by,

$$\mathcal{H} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \mu\Phi\phi^2,$$

we are to determine the lifetime of a Φ particle to decay into two ϕ's to the lowest order in μ assuming that $M > 2m$.

We first notice that the interaction Hamiltonian is $\int d^3x \mu\Phi\phi\phi$. From this, we can directly calculate the amplitude associated with our desired diagram:

$i\mathcal{M} =$



$= -2i\mu,$

The factor of 2 comes from Bose statistics associated with the two identical final ϕ particles. So,

$$|\mathcal{M}|^2 = 4\mu^2.$$

We have shown before that we can directly compute the decay width of a particle from the amplitude by using the relation,

$$\Gamma = \frac{1}{2M}\int \frac{d\Omega}{16\pi^2}\frac{|\vec{k}|}{E_{cm}}|\mathcal{M}|^2.$$

In the center of mass frame, the rest frame of the Φ, $E_{cm} = M$, $p = (M, \vec{0}), k_1 = (M/2, \vec{k})$, and $k_2 = (M/2, -\vec{k})$. From simple kinematics it is clear that $|\vec{k}| = \left(\frac{M^2}{4} - m^2\right)^{1/2} = \frac{M}{2}\left(1 - 4\frac{m^2}{M^2}\right)^{1/2}$. This leads to

$$\Gamma = \frac{4\mu^2 M^2}{64\pi^2 M^2}\left(1 - 4\frac{m^2}{M^2}\right)^{1/2}\int d\Omega.$$

When we integrate over the solid angle Ω, we should only cover 2π because the ϕ's are identical. After integrating and simplifying terms we find that

$$\Gamma = \frac{\mu^2}{8\pi M}\left(1 - 4\frac{m^2}{M^2}\right)^{1/2}. \tag{1.1}$$

$$\boxed{\therefore \tau = \frac{8\pi M}{\mu^2}\left(1 - 4\frac{m^2}{M^2}\right)^{-1/2}.} \tag{1.2}$$

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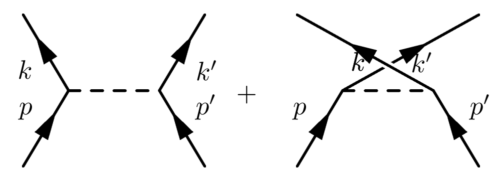
2. Massless Fermion Scattering in Yukawa Theory

a) We are to write the complete amplitude for scattering two massless fermions in Yukawa theory. From previous homework and class notes this is,

$i\mathcal{M} =$



$$= (-ig^2)\left(\bar{u}(k)u(p)\frac{1}{(k-p)^2 - m_\phi^2}\bar{u}(k')u(p') - \bar{u}(k)u(p')\frac{1}{(p-k')^2 - m_\phi^2}\bar{u}(k')u(p)\right). \tag{2.1}$$

b) We are to compute the spin-averaged square of this amplitude explicitly. We will make explicit use of our trace identities and will simplify in terms of the standard Mandelstam variables s, t and u.

Let us begin our derivation by noting that the Mandelstam variables (in the massless limit) are given by

$$s = (p + p')^2 = (k + k')^2 = 2p \cdot p' = 2k \cdot k';$$
$$t = (k - p)^2 = (k' - p')^2 = -2p \cdot k = -2p' \cdot k';$$
$$u = (k' - p)^2 = (k - p')^2 = -2p \cdot k' = -2p' \cdot k;$$

We can now directly compute the spin averaged squared amplitude. When using the standard trace technology, we will simplify our terms by noticing that $m_f = 0$.

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{g^4}{4}\sum_{\text{spin}}\Bigg\{\frac{1}{(t-m_\phi^2)^2}\bar{u}(k)u(p)\bar{u}(p)u(k)\bar{u}(k')u(p')\bar{u}(p')u(k') \\
&\qquad + \frac{1}{(u-m_\phi^2)^2}\bar{u}(k)u(p')\bar{u}(p')u(k)\bar{u}(k')u(p)\bar{u}(p)u(k') \\
&\qquad - \frac{2}{(t-m_\phi^2)(u-m_\phi^2)}\bar{u}(k)u(p')\bar{u}(k')u(p)\bar{u}(p')u(k')\bar{u}(p)u(k)\Bigg\}, \\
&= \frac{g^4}{4}\left\{\frac{1}{(t-m_\phi^2)^2}\text{Tr}\,[\not{p}\not{k}]\,\text{Tr}\,[\not{p}'\not{k}'] + \frac{1}{(u-m_\phi^2)^2}\text{Tr}\,[\not{p}\not{k}']\,\text{Tr}\,[\not{p}'\not{k}] - \frac{2}{(t-m_\phi^2)(u-m_\phi^2)}\text{Tr}\,[\not{k}\not{p}'\not{k}'\not{p}]\right\}, \\
&= \frac{g^4}{4}\Bigg\{\frac{16(p\cdot k)(p\cdot k)}{(t-m_\phi^2)^2} + \frac{16(p\cdot k')(p'\cdot k)}{(u-m_\phi^2)^2} \\
&\qquad - \frac{8}{(t-m_\phi^2)(u-m_\phi^2)}\Big((k\cdot p)(k'\cdot p') + (p'\cdot k)(p\cdot k') - (p\cdot p')(k\cdot k')\Big)\Bigg\}, \\
&= \frac{g^4}{4}\left\{\frac{4t^2}{(t-m_\phi^2)^2} + \frac{4u^2}{(u-m_\phi^2)^2} - \frac{8}{(t-m_\phi^2)(u-m_\phi^2)}\left(\frac{t^2}{4}+\frac{u^2}{4}-\frac{s^2}{4}\right)\right\}, \\
&= g^4\left\{\frac{t^2}{(t-m_\phi^2)^2} + \frac{u^2}{(u-m_\phi^2)^2} - \frac{t^2+u^2-s^2}{2(t-m_\phi^2)(u-m_\phi^2)}\right\}.
\end{aligned}$$

We can simplify this equation by recalling that, in general, $\sum_i m_i = s + t + u$. In the massless case this reduces to $s + t + u = 0$ and so $s^2 = -(t + u)^2$. We may therefore conclude that

$$\boxed{\therefore \overline{|\mathcal{M}|^2} = g^4\left\{\frac{t^2}{(t-m_\phi^2)^2} + \frac{u^2}{(u-m_\phi^2)^2} + \frac{tu}{(t-m_\phi^2)(u-m_\phi^2)}\right\}.} \tag{2.2}$$

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c) Let us reduce equation (2.2) to the case where $m_\phi = 0$. By sight, this becomes

$$\boxed{\overline{|\mathcal{M}|^2} = g^4(1 + 1 + 1) = 3g^4.} \tag{2.3}$$

It is worth noting that this agrees with our homework result.

d) Let us now compute the total cross section for this event. We have previously demonstrated that in the center of mass frame the differential cross section is given by

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 E_{cm}^2}.$$

To determine the total cross section, we must integrate over half the solid angle giving us a factor of 2π.

$$\boxed{\therefore \sigma = \frac{3g^4}{32\pi E_{cm}^2}.} \tag{2.4}$$

3. The Ward Identity for Compton Scattering

We are to explicitly verify the Ward identity, $k_\nu\mathcal{M}^\nu = 0$, for the case of Compton scattering. This is equivalent to a demonstration that when $\epsilon_\nu(k) = k_\nu$,

$$i\mathcal{M} = -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} - \frac{2\gamma^\nu p^\mu - \gamma^\nu \not{k}'\gamma^\mu}{2p\cdot k'}\right]u(p) = 0.$$

This demonstration will be much clearer if we rewrite the second term in the amplitude in terms of $(\not{p}' - \not{k})$ instead of $(\not{p} - \not{k}')$. This is reasonable because by momentum conservation $p - k' = p' - k$. To rewrite the amplitude, however, it is important to notice that the contraction that was used for simplification, $(\not{p} + m)\gamma^\mu u(p) = 2p^\mu u(p)$ cannot be used when we use $(\not{p}' - \not{k} + m)$. We can, however, contract to the left using $\bar{u}(p')$. Doing so will yield

$$i\mathcal{M} = -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} - \frac{2p'^\nu\gamma^\mu - \gamma^\nu \not{k}\gamma^\mu}{2p'\cdot k}\right]u(p).$$

Let us derive this amplitude for the case of $\epsilon_\nu(k) = k_\nu$ by brute force.

$$\begin{aligned}
i\mathcal{M} &= -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} - \frac{2p'^\nu\gamma^\mu - \gamma^\nu \not{k}\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2\epsilon^*_\mu(k')\bar{u}(p')\left[k_\nu\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} - k_\nu\frac{2p'^\nu\gamma^\mu - \gamma^\nu \not{k}\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2\epsilon^*_\mu(k')\bar{u}(p')\left[\frac{k_\nu\gamma^\mu \not{k}\gamma^\nu + 2p\cdot k\gamma^\mu}{2p\cdot k} - \frac{2p'\cdot k\gamma^\mu - k_\nu\gamma^\nu \not{k}\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2\epsilon^*_\mu(k')\bar{u}(p')\left[\frac{k_\nu\gamma^\mu k_\rho\gamma^\rho\gamma^\nu + 2p\cdot k\gamma^\mu}{2p\cdot k} - \frac{2p'\cdot k\gamma^\mu - \not{k}\not{k}\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2\epsilon^*_\mu(k')\bar{u}(p')\left[\frac{2\not{k}k^\mu - 2k^\mu\not{k} + 2k^2\gamma^\mu - \not{k}^2\gamma^\mu + 2p\cdot k\gamma^\mu}{2p\cdot k} - \frac{2p'\cdot k\gamma^\mu - \not{k}^2\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2\epsilon^*_\mu(k')\bar{u}(p')\left[\frac{2p\cdot k\gamma^\mu}{2p\cdot k} - \frac{2p'\cdot k\gamma^\mu}{2p'\cdot k}\right]u(p),\\
&= -ie^2\epsilon^*_\mu(k')\bar{u}(p')\left[\gamma^\mu - \gamma^\mu\right]u(p) = 0.
\end{aligned}$$

$$\boxed{\therefore k_\nu\mathcal{M}^\nu(k) = 0.} \qquad (3.1)$$

4. Compton Scattering in Scalar Quantum Electrodynamics

We are to consider the physics governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2.$$

As usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi \equiv \partial_\mu\phi + ieA_\mu\phi$.

a) The Lagrangian is clearly invariant under the transformation $\phi \to e^{-ie\alpha}\phi$ because it contains only squared terms and we can assume for now that α is a constant. So $\mathcal{L} \to \mathcal{L}' = \mathcal{L}$. Let us compute the conserved Noether current.

First, let us rewrite the global phase transition to the first order to determine the variation on each of the complex fields.

$$\phi \to \phi' = e^{-ie\alpha}\phi \approx (1 - ie\alpha)\phi \Rightarrow \Delta\phi = -ie\phi;$$
$$\phi^\dagger \to \phi'^\dagger = e^{ie\alpha}\phi^\dagger \approx (1 + ie\alpha)\phi^\dagger \Rightarrow \Delta\phi^\dagger = ie\phi^\dagger.$$

We can use this to calculate the conserved Noether current associated with this symmetry. From our earlier work in class and homework, we know that,

$$\begin{aligned}
j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\Delta\phi^\dagger,\\
&= \left((\partial^\mu\phi^\dagger - ieA^\mu\phi^\dagger)(-ie\phi) + (\partial^\mu\phi + ieA^\mu\phi)(ie\phi^\dagger)\right),\\
&= \left((-ie\phi)D^\mu\phi^\dagger + (ie\phi^\dagger)D^\mu\phi\right),
\end{aligned}$$

$$\boxed{\therefore j^\mu = ie\left(\phi^\dagger D^\mu\phi - \phi D^\mu\phi^\dagger\right).} \qquad (4.1)$$

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b) Even more interesting than global phase invariance, however, is that the Lagrangian is in fact locally gauge invariant. A transformation of the form $\phi \to e^{-ie\alpha(x)}\phi$ will leave the Lagrangian unchanged. The field strength tensor is invariant to this gauge as we know from electrodynamics. Let us consider how the covariant derivative and the vector potential must transform to preserve invariance with respect to this gauge.

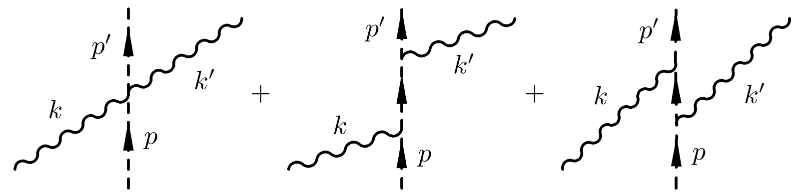
By direct calculation, we see that

$$D_\mu\phi \to D_\mu^\alpha = e^{-ie\alpha(x)}D_\mu\phi - ie\partial_\mu\alpha(x)e^{-ie\alpha(x)}\phi.$$

We can transform the vector potential by $A_\mu \to A_\mu^\alpha = A_\mu + \partial_\mu\alpha(x)$, and leave $F_{\mu\nu}$ invariant because we only add a total derivative. By adding this term, however, D_μ will become invariant under the local phase transformation. For precisely this utility, A_μ is defined to transform in just the right way to maintain D_μ's covariance. So,

$$A_\mu \to A_\mu^\alpha = A_\mu + \partial_\mu\alpha(x).$$

c) We are to draw the Feynman diagrams for $\gamma\phi^- \to \gamma\phi^-$ in scalar quantum electrodynamics to the order e^2. Using our given vertex terms and propagator terms derived earlier, we may directly write the diagram. They are all additive by Bose statistics.



d) The amplitude for this interaction is,

$$i\mathcal{M} = \Big\{\epsilon'^*_\mu(k')2ie^2g^{\mu\nu}\epsilon_\nu(k) + \epsilon'^*_\mu(k')(-ie(p+p'+k)^\mu)\frac{i}{(p+k)^2-m^2}(-ie(p+p'+k')^\nu)\epsilon_\nu(k)$$
$$+\epsilon'^*_\mu(k')(-ie(p+p'-k)^\mu)\frac{i}{(p-k')^2-m^2}(-i(p+p'-k')^\nu)\epsilon_\nu(k)\Big\},$$
$$= -ie^2\epsilon'^*_\mu(k')\epsilon_\nu(k)\left\{-2g^{\mu\nu} + \frac{(p+p'+k')^\nu(p+p'+k)^\mu}{2p\cdot k} - \frac{(p+p'-k')^\nu(p+p'-k)^\mu}{2p'\cdot k}\right\},$$
$$= -ie^2\epsilon'^*_\mu(k')\epsilon_\nu(k)\left\{-2g^{\mu\nu} + \frac{(2p+k)^\nu(2p'+k')^\mu}{2p\cdot k} - \frac{(2p'-k)^\nu(2p-k')^\mu}{2p'\cdot k}\right\}.$$

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e) As in question (3) above, we must explicitly demonstrate the result of the Ward identity. This can be accomplished by setting $\epsilon_\nu(k) = k_\nu$ in the equation for the amplitude and see that $\mathcal{M} \to 0$.

To demonstrate this case, it will be helpful to recall that a photon is represented by a null vector, $k_\nu k^\nu = 0$, and that momentum is conserved, $p + k - p' - k' = 0$. Let us derive the result.

$$i\mathcal{M} = -ie^2\epsilon'^*_\mu(k')k_\nu\left\{-2g^{\mu\nu} + \frac{(2p+k)^\nu(2p'+k')^\mu}{2p\cdot k} - \frac{(2p'-k)^\nu(2p-k')^\mu}{2p'\cdot k}\right\},$$
$$= -ie^2\epsilon'^*_\mu(k')\left\{-2k^\mu + \frac{k_\nu(2p+k)^\nu(2p'+k')^\mu}{2p\cdot k} - \frac{k_\nu(2p'-k)^\nu(2p-k')^\mu}{2p'\cdot k}\right\},$$
$$= -ie^2\epsilon'^*_\mu(k')\left\{-2k^\mu + \frac{(2p\cdot k)(2p'+k')^\mu}{2p\cdot k} - \frac{(2p'\cdot k)(2p-k')^\mu}{2p'\cdot k}\right\},$$
$$= -ie^2\epsilon'^*_\mu(k')\left\{-2k^\mu + (2p'+k')^\mu - (2p-k')^\mu\right\},$$
$$= -2ie^2\epsilon'^*_\mu(k')\left\{-p^\mu - k^\mu + p'^\mu + k'^\mu\right\},$$
$$= -2ie^2\epsilon'^*_\mu(k')\left\{0\right\}, = 0.$$

$$\boxed{\therefore k_\nu\mathcal{M}^\nu(k) = 0.} \tag{4.2}$$

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